Building bridges between geometry and algebra

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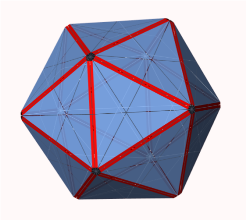
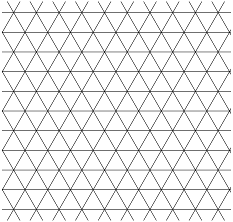
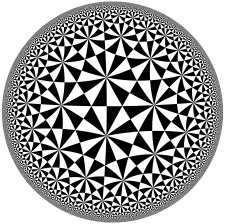
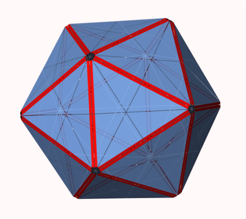




Coxeter groups

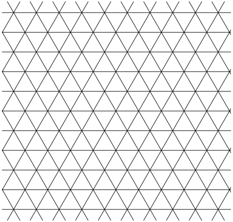
Coxeter groups are abstract versions of reflection groups that admit a finite presentations as

$$W = \langle s_i, s_j \in S \mid s_i^2, (s_i s_j)^{m_{i,j}} \rangle, \text{ where } 2 \leq m_{i,j} \leq \infty$$

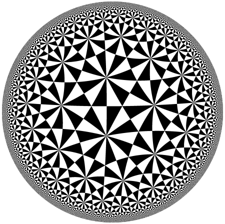
= ∞ means no relation



W_1



W_2

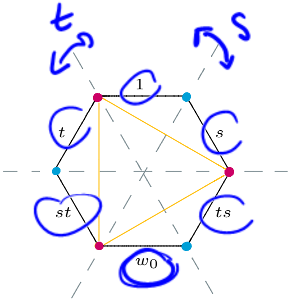


W_3

$$W_1 = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^2, (s_2 s_3)^5, (s_3 s_1)^3 \rangle$$
$$W_2 = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^3, (s_2 s_3)^3, (s_3 s_1)^3 \rangle$$
$$W_3 = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^2, (s_2 s_3)^3, (s_3 s_1)^7 \rangle$$

Spherical example

Finite Coxeter groups act on simplicial spheres.



Coxeterkomplex of $W_0 \cong S_3$

$\langle s, t \mid s^2, t^2, (st)^3 \rangle$

$sts = tst$

Spherical example

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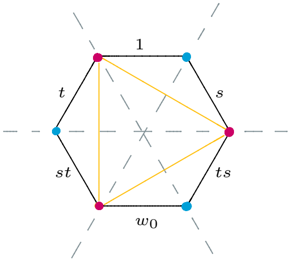
The following properties hold:



Coxeterkomplex of $W_0 \cong S_3$

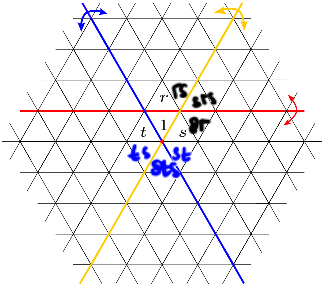
- maximal simplices correspond (1:1) to elements of W_0
- smaller simplices can be colored using (subsets of the) generators
- there exists a unique longest element in W_0 $w_0 = sts = tst$.

$$W_0 = \langle s, t \mid s^2, t^2, (st)^3 \rangle$$

Affine example

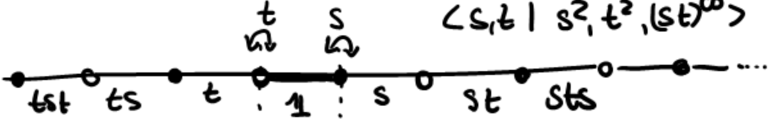
Affine Coxeter groups act on simplicial (tiled) $\mathbb{R}^n$.

$\mathbb{R}^2$



Coxeterkomplex of $W_0 \cong S_3$

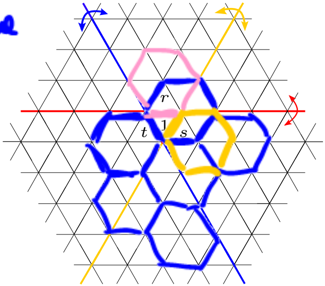
$\mathbb{R}^1$



Affine example

Affine Coxeter groups act on simplicial (tiled) $\mathbb{R}^n$.
The following properties hold:

t=blue



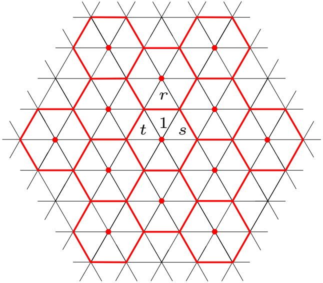
Coxeterkomplex of $W_0 \cong S_3$

- maximal simplices correspond (1:1) to elements of W
- smaller simplices can be colored using (subsets of the) generators
- each element $w \in W$ has a unique normal form $w = t_\lambda u$ and $W = W_0 \ltimes T$.

$$W_0 = \langle r, s, t \mid r^2, s^2, t^2, (xy)^3 \text{ for } x \neq y \in \{r, s, t\} \rangle$$

Affine example

Every infinite Coxeter group acts on a simplicial (tiled) $\mathbb{R}^n$.
The following properties hold:



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Outline of the talk

- An easy to state (but not fully solved) problem
- Beautiful combinatorics and geometry
- Why one should care

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- An easy to state (but not fully solved) problem
  Computing reflection length
- Beautiful combinatorics and geometry
  Folded galleries and their shadows
- Why one should care
  Computing dimensions of ADL-varieties

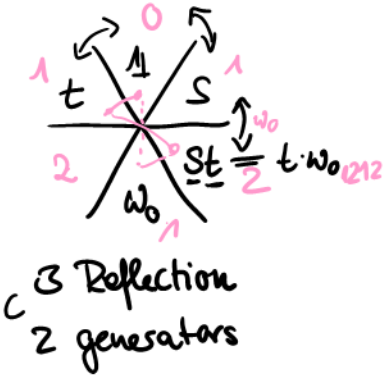
Computing reflection length

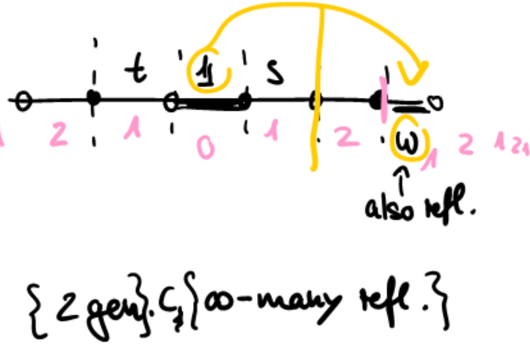
Reflection length $l_R(w)$

The set $R = \bigcup_{w \in W} wSw^{-1}$ of all reflections in a Coxeter group is a natural generating set.

Question

Given a fixed $w \in W$ what is the smallest number $k \in \mathbb{N}$ such that w is the product of k reflections?





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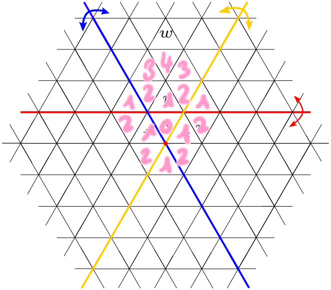


4 uniform upper bound

in general

$l_R(w) \le 2n$

$w \in W \curvearrowright \mathbb{R}^n$

Reflection length $l_R(w)$

Question

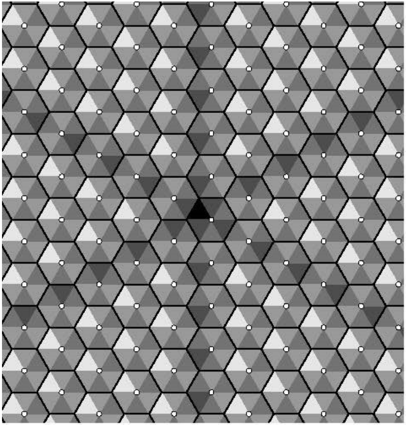
Given $w \in W$ what is the smallest $k \in \mathbb{N}$ such that $l_R(w) = k$?

W finite/affine

- l_R is uniformly bounded
- explicit formulas and characterizations.

(Carter '72,

Lewis-McCammond-Petersen-Schwer '18)



Reflection length $l_R(w)$

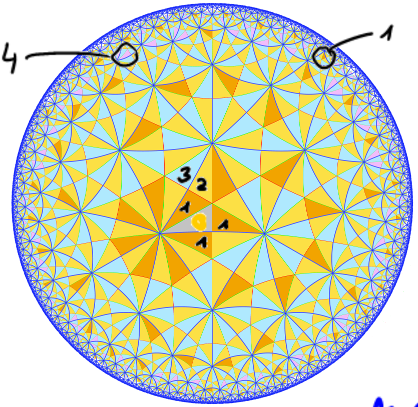
Question

Given $w \in W$ what is the smallest $k \in \mathbb{N}$ such that $l_R(w) = k$?

W hyperbolic:

- reflection length is unbounded
- more structure is known for universal Coxeter groups.

(Duszenko '12, Drake-Peters/Lotz '21).



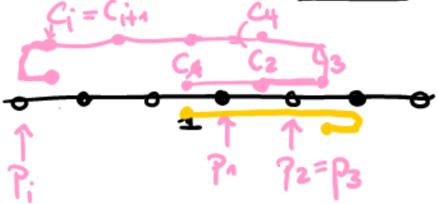
same color means same refl. length

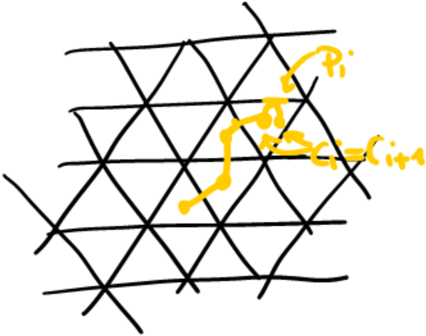
Folded galleries and their shadows

Galleries in Coxeter Complexes

Definition

A *gallery* γ in a Coxeter complex is a sequence of maximal simplices c_i such that subsequent ones share a codimension one face p_i. We write $\gamma = (c_1, p_1, c_2, p_2, \ldots, p_{n-1}, c_n)$. We say a gallery is *folded* at i if $c_i = c_{i+1}$.





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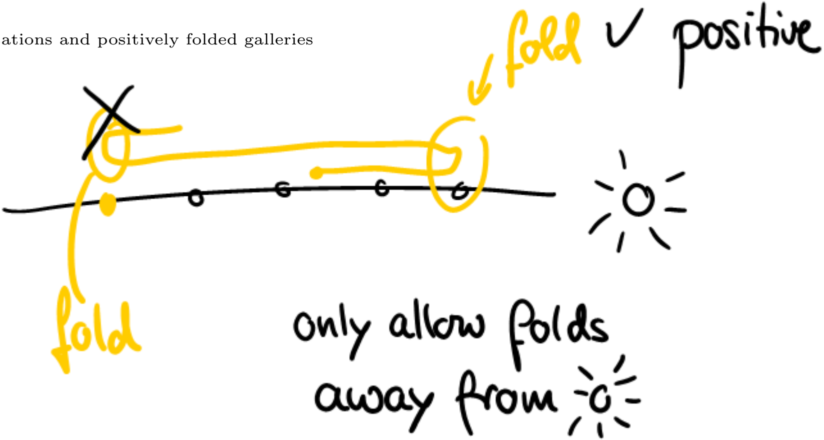
- The p_i's are colored by generators s_i and determine a word, calles the *type* of γ, representing an element in W. We call this word the *type* of γ.
- Any word determines a unique (unfolded) gallery starting in the identity simplex $c_1 = 1$.
- Folded galleries starting in 1 correspond to 'decorated' words.

Positive folds

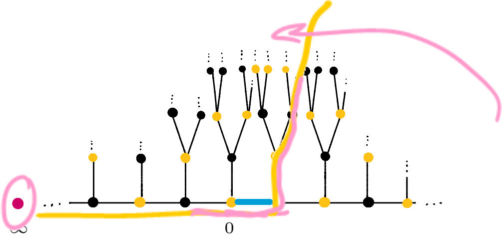
orientations and positively folded galleries



Folded galleries arise naturally via retractions in buildings.

Affine buildings in dimension one

Trees without leafs are exactly the 1-dim. affine buildings.



affine Bruhat-Tits building for $\mathrm{SL}_2(\mathbb{Q}_2)$ with 2-adic valuation

$G(F)/I$

$I = \mathrm{Stab}(\text{blue edge})$

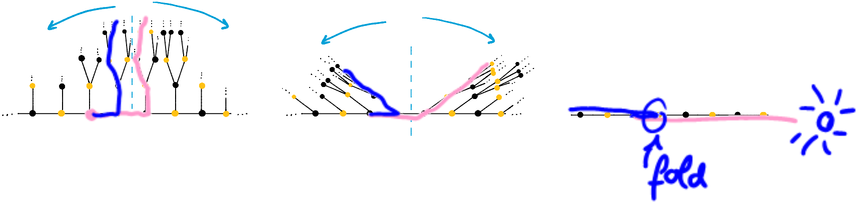
$$\mathrm{SL}_2(\mathbb{Q}_2) = UTK$$

IgI

- $U = \{\left(\begin{smallmatrix} 1 & 0 \\ \star & 1 \end{smallmatrix}\right)\} \mathrel{\widehat{=}} \mathrm{Stab}(\infty)$
- $T = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\} \cong \mathbb{Z} \mathrel{\widehat{=}}$ horizontal translations
- $K = \{\left(\begin{smallmatrix} c & s \\ -s & c \end{smallmatrix}\right) \mid c^2 + s^2 = 1\} = G(\mathcal{O}) \mathrel{\widehat{=}} \mathrm{Stab}(0)$

Retractions

Retraction $r : X \to A$ based at a maximal simplex.
For all $t \in T$ one has $r^{-1}(W_0.t) = K.t$ and
$r^{-1}(t) = I.t$, where $I := \mathrm{Stab}(\mathbb{1})$.



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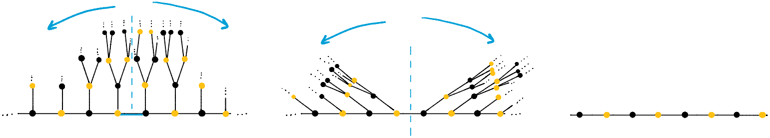
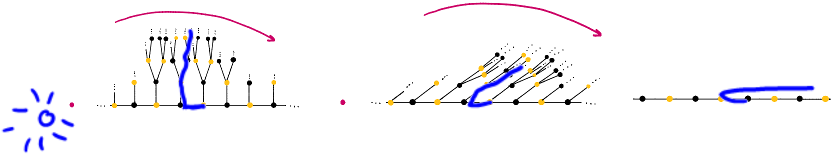


Retraction $\rho : X \to A$ based at a direction at infinity.
For all $t \in T$ one has $\rho^{-1}(t) = U.t$.



□

Shadows of elements in W

Definition

The *shadow* $\mathrm{Sh}_\phi(w)$ of $w \in W$ with respect to an orientation ϕ is the set of all ends of positively folded galleries of type w.

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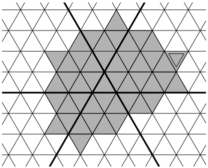

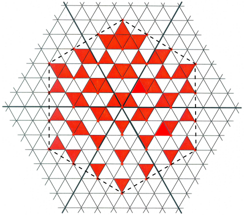

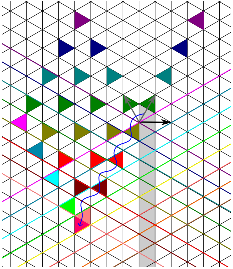

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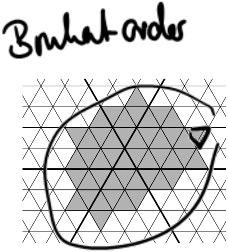
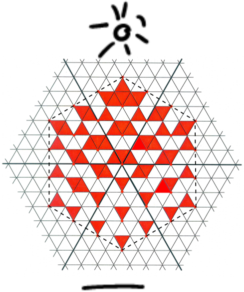
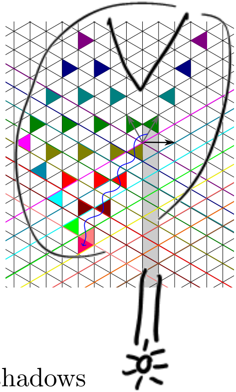






One has:

- [GS18] Recursive descriptions of alcove shadows
- [MNST19] Recursive description of shadows with respect to general chimneys + geometric interpretations in terms of retractions in a building.

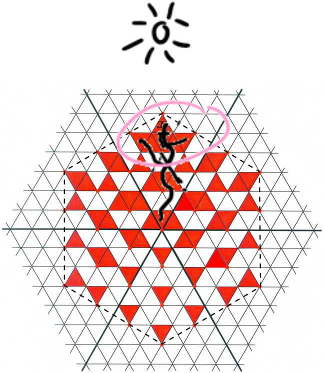
Computing dimensions of affine Deligne-Lustig varieties

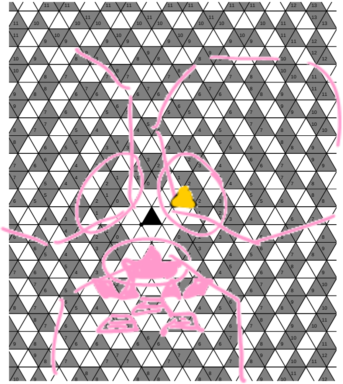
Why one should care

Shadows and folded galleries explain many algebraic situations. For example

- [Gaussent-Littelmann] Study highest weight representations of complex semi-simple algebraic groups using LS–galleries.
- [Ehrig] Description of MV polytopes via LS–galleries and retractions.
- [Schwer=Hitzelberger] Schur-Horn type theorems for algebraic groups over non-archimedean local fields with valuation.
- [Milićević–Schwer–Thomas] Proof of nonemptiness and dimensions of (certain) ADLVs.

Folded galleries and ADLV.





Red alcoves are ends of positively folded galleries of fixed type x.
Grey alcoves y are non-empty ADLV $X_y(t^\rho)$.

Picture on the right: Görtz–Haines–Kottwitz–Reuman, arXiv:0504443

$X_w(b)$
elts in Cox grp

Affine flag variety

- $\mathbb{F}_q$ a finite field of order $q = p^m$, σ its Frobenius
- $F = k((t))$ where $k = \overline{\mathbb{F}}_q$ (non-archimedian local field)
- $\mathcal{O} = k[[t]]$ (ring of integers)
- project $\mathcal{O} \to k$ by setting $t = 0$, detects constant term a_0

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- project $\mathcal{O} \to k$ by setting $t = 0$, detects constant term a_0

The *affine flag variety* is the quotient $G(F)/I$, where

- G is a connected reductive group over $\mathbb{F}_q$,
- $B \subset G$ a Borel subgroup containing a maximal torus T and
- I denotes the *Iwahori subgroup* of $G(F)$ that is the inverse image of $B(k)$ under the projection $G(\mathcal{O}) \to G(k)$.

The points of the affine flag variety correspond to the maximal simplices in a Bruhat-Tits building.

Stab of the edge labeled 1

Definition of ADLVs

G connected reductive group over $\mathbb{F}_q$
I Iwahori subgroup
W affine Weyl group
$k = \overline{\mathbb{F}}_q$, $F = k((t))$, σ the Frobenius map
$G(F) = \bigsqcup_{x \in W} IxI$ Iwahori-Bruhat decomposition

Definition

The *affine Deligne–Lusztig variety* $X_x(b) \subseteq G(F)/I$ is given by

$$X_x(b) = \{g \in G(F) \mid g^{-1} b \sigma(g) \in IxI\}/I,$$

where $x \in W, b \in G(F)$.

Main Questions

Nonemptiness: For which $(x, b) \in W \times W$ is $X_x(b) \neq \emptyset$?

Dimension: What is the dimension of $X_x(b)$?

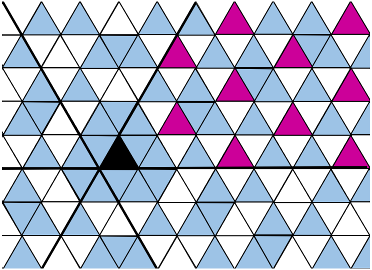
In case b is *basic* these questions are solved:

- Görtz, Haines, He, Kottwitz, Milićević (nèe Beazley), Reuman, Viehmann, ...
- Görtz, He and Nie (2012):
  nonemptiness pattern for all x and all basic b
- He (Annals 2014):
  dimension formula for all x and basic b

The *basic* case

An element $b \in G(F)$ is *basic* if it is σ-conjugate to an element of length 0 in the extended affine Weyl group.

- All basic b in W are pairwise σ-conjugate.
- Dominant translations (pink) are not basic and pairwise not σ-conjugate.



basic elements (blue); translations in the dominant Weyl chamber (pink)

A new geometric approach

In the following let $b = t^\lambda$ be a translation in W.
We proceed as follows:

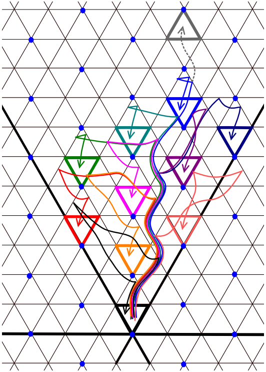
(1) $X_x(b) \neq \emptyset \iff$ there exists a positively folded gallery from 1 to b of type x.

(2) $dim(X_x(b))$ can be computed via positive folds + positive crossings of these galleries

(3) Construct and manipulate such galleries using root operators, combinatorics in Coxeter complexes and explicit transformations.

[MST19,20], [MNST20]

Manipulations using root operators



Applying available root operators to explicitly constructed galleries we obtain: $X_x(b) \neq \emptyset$ for most $b = t^\mu$ between 1 and x.

Theorem (Milićević–S–Thomas, 2019)

Let $b = t^\mu$ be a pure translation and let $x = t^\lambda w \in W$. Assume that b is in the convex hull of x and the base alcove + two technical conditions on μ and λ. Then

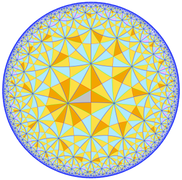
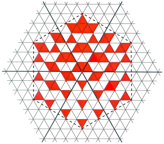
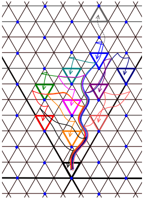
$$X_x(1) \neq \emptyset \implies X_x(b) \neq \emptyset$$

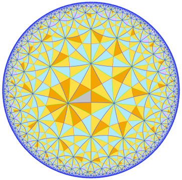
and if $w = w_0$ then $X_x(1) \neq \emptyset$ and $X_x(b) \neq \emptyset$. If both varieties are nonempty then

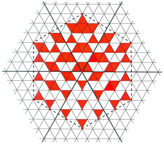
$$\dim X_x(b) = \dim X_x(1) - \langle \rho, \mu^+ \rangle.$$

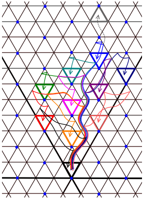
Precise assumptions:

- $t^\lambda w_0$ and $t^{-\mu} x$ are in the shrunken dominant Weyl chamber $\widetilde{\mathcal{C}}_f$
- b is in the convex hull of x and the base alcove
- μ lies in the negative cone based at $\lambda - 2\rho$.







Thank you!

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